Midterm Exam

EXAM INFORMATION

General Note: In our view, the most important issue is to know how to address a particular problem. Therefore, there will be partial credit for good solution outlines even if not all the mathematical manipulations are completed.

- This exam is open-book and open-notes. You are allowed to access any of the course materials provided to you (problem set solutions, lecture notes, lecture slides). However, you may wish to make a sheet of notes to have the main formulas handy.
- You have 90 minutes to complete this exam and 45 minutes of technical time to submit your answers. The idea is that the exam is over by 3:45pm and you have until 4:30 to scan and submit everything. You may also scan and submit your solutions as you do each problem and you are allowed to resubmit your answers more than once.
- In case of questions Teaching Assistants and the Professor are available on the course Discord Server. We prefer that you use the discord server. However, in case you have problems with the discord server you could also try Zoom, or you may call or email the Professor (yanina.shkel@epfl.ch and +41 21 693 13 43).
- Grading of True/False Questions (Problem 1): For each sub-problem, your score will depend on the correctness of the answer as well as a brief justification provided.
- For Problems 2, 3, 4 and 5: Unless explicitly stated otherwise, detailed derivations of the results are required for full credit.
- Label your files as LastnameFirstnameProblemNumber and convert them to a pdf format. So, for example, if the Professor submitted Problem 1, she would write ShkeYanianProblem1.pdf
- Submit the scan of the answers separately for each problem in the provided Moodle dropbox (5 dropboxes total).

ACADEMIC INTEGRITY

- All answers should be handwritten in your own handwriting. Notepad devices like ipads are also allowed for writing down the answers.
- Calculators, computing and communication devices ARE NOT permitted to be used to answer the exam questions.
- You ARE NOT allowed to discuss the exam with anyone in the course or outside of the course for the duration of the exam and in particular all work must be your own. The only exception to this rule is the Graduate Teaching Assistants and the Professor.
- You ARE NOT allowed to google answers.
- In case we suspect academic misconduct, we will investigate and pursue disciplinary action. For example, we reserve the right to question select students about their answers after the exam in order to confirm that the answers provided were indeed their own.

*** Good Luck! ***

Problem 1 *(True/False Questions)* 24 points

For each statement specify if it is true of false. Provide a short (about two lines) justification.

(a) (4 Pts) True/False: Let $Y(e^{j\omega})$ denote the Fourier transform of $y[n]$. Then, the Fourier transform of $y[n-n_0]+y[-n]$ is $e^{-n_0 j\omega}Y(e^{j\omega})-Y(e^{-j\omega})$.

Solution: False, the Fourier transform of $y[n-n_0]+y[-n]$ is $e^{-n_0 j\omega}Y(e^{j\omega})+Y(e^{-j\omega})$.

Note that the original question had a typo and the signal was given as $y_1[n-n_0]+y_2[-n]$. So, we also accept it as correct answer if students thought this was a trick question and said False for that reason.

The correction on Moodle was given as $y[n-n_0]-y[-n]$. So, we also accept it as a correct answer if students said true because the Fourier Transform of this signal matches the one given.

(b) (4 Pts) True/False: The system $y[n]=nx[n]$ is invertible.

Solution: False, both $x[n]=\delta[n]$ and $x[n]=2\delta[n]$ give the same output.

(c) (4 Pts) True/False: If $x(t)=\sqrt{\frac{2}{3}}\text{sinc}\left(\frac{2}{3}t\right)$ then $X(\omega)=0$ for $|\omega|>\frac{2\pi}{3}$.

Solution: True, $X(\omega)$ is simply a box function with limits at $|\omega|=\frac{2\pi}{3}$.

(d) (4 Pts) True/False: If $y[n]=x[n]*h[n]$ then $y[-n]=x[-n]*h[-n]$.

Solution: True, since $x[-n]*h[-n]=\sum_{k=-\infty}^{\infty}x[-k]h[k-n]=\sum_{\tilde{k}=-\infty}^{\infty}x[\tilde{k}]h[-\tilde{k}-n]=y[-n]$.
You could also check that this is true by applying the DTFT together with the time reversal property to $h[-n]$ and $x[-n]$.

(e) (4 Pts) True/False: If $y(t)=x(t)*h(t)$ then $y(t-4)=x(t-2)*h(t-2)$.

Solution: True, we can write $x(t-2)*h(t-2)=(x(t)*\delta(t-2))*(h(t)*\delta(t-2))=(x(t)*h(t))*(\delta(t-2)*\delta(t-2))=y(t-4)$.

(f) (4 Pts) True/False: The signal $x[n]=\cos 3n$ is periodic.

Solution: False, since the equation $3N=2\pi$ does not have an integer solution (see PSET 1, Problem 3).

Problem 2 *(Difference Equations.)* 12 points

A stable LTI system is described by the following difference equation:

$$8y[n] + 2y[n-1] - y[n-2] = 8x[n]$$

(a) (6 Pts) Find the frequency response of this system.

(b) (6 Pts) Find the impulse response of this system.

Solution: (a)

By using time-shift property together with linearity of DTFT we obtain

$$Y(e^{j\omega})(8 + 2e^{-j\omega} - e^{-2j\omega}) = 8X(e^{j\omega}) \tag{1}$$

Then, using the expression $Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$ for the frequency response

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{8}{8 + 2e^{-j\omega} - e^{-2j\omega}} \tag{2}$$

$$= \frac{8}{(2 + e^{-j\omega})(4 - e^{-j\omega})} = \frac{2}{3}\frac{1}{1 + \frac{1}{2}e^{-j\omega}} + \frac{1}{3}\frac{1}{1 - \frac{1}{4}e^{-j\omega}} \tag{3}$$

The last step is obtained by substituting $x = e^{-j\omega}$ and computing a partial fraction expansion for

$$\frac{8}{8 + 2x - x^2} = \frac{A}{2 + x} + \frac{B}{4 - x}. \tag{4}$$

(b)

Using the inverse DTFT,

$$h[n] = \frac{2}{3}\left(-\frac{1}{2}\right)^n u[n] + \frac{1}{3}\left(\frac{1}{4}\right)^n u[n] \tag{5}$$

Problem 3 *(Differential Equations.)* 20 points

(a) (6 Pts) Find the frequency response of a stable LTI system characterized by

$$\frac{d^2}{dt^2}y(t) + 11\frac{d}{dt}y(t) + 10y(t) = 9x(t).$$

(b) (6 Pts) Find the impulse response of this system.

(c) (8 Pts) Suppose that an input $x(t) = \frac{1}{3}e^{-t}u(t)$ is applied to this system. What is the output $y(t)$?

Solution:

(a)

Applying the Fourier Transform with differentiation in time and linearity properties gives

$$(j\omega)^2 Y(\omega) + 11j\omega Y(\omega) + 10Y(\omega) = 9X(\omega).$$

Rearranging the terms we obtain

$$Y(\omega) = \frac{9}{(j\omega)^2 + 11j\omega + 10} X(\omega).$$

Since $Y(\omega) = H(\omega)X(\omega)$, the frequency response is

$$H(\omega) = \frac{9}{(j\omega)^2 + 11j\omega + 10}.$$

(b)

We can rewrite the frequency response as

$$H(\omega) = \frac{9}{(j\omega)^2 + 11j\omega + 10} = \frac{9}{(j\omega + 10)(j\omega + 1)} = \frac{1}{j\omega + 1} - \frac{1}{j\omega + 10}.$$

The last equality follows by applying the partial fraction expansion procedure and solving for A and B in the equation

$$\frac{9}{x^2 + 11x + 10} = \frac{A}{x+1} + \frac{B}{x+10}.$$

Applying the inverse Fourier transform (one-sided exponential pair in Appendix 4.B) to each term in the expression for $H(\omega)$ gives the impulse response

$$h(t) = e^{-t}u(t) - e^{-10t}u(t).$$

(c)

This problem could be solved in the time domain using convolution. However, it may be easier to solve it in the frequency domain. The Fourier Transform of $x(t) = \frac{1}{3}e^{-t}u(t)$ is (from Appendix 4.B)

$$X(\omega) = \frac{1}{3}\frac{1}{1 + j\omega}$$

Then

$$Y(\omega) = \frac{9}{(j\omega)^2 + 11j\omega + 10} X(\omega) = \frac{3}{(j\omega + 10)(j\omega + 1)^2}.$$

To invert $Y(\omega)$ we again need to apply the partial fraction expansion. That is

$$Y(\omega) = \frac{A}{(j\omega + 1)^2} + \frac{B}{j\omega + 1} + \frac{C}{j\omega + 10} = \frac{1}{3}\frac{1}{(j\omega + 1)^2} - \frac{1}{27}\frac{B}{j\omega + 1} + \frac{1}{27}\frac{1}{j\omega + 10}.$$

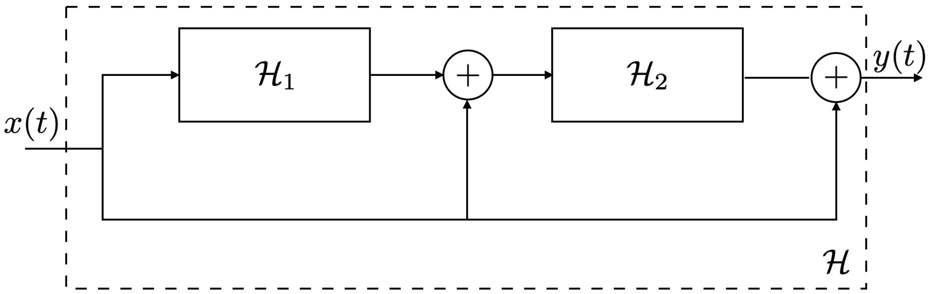
Applying the inverse Fourier transform (one-sided exponential pair in Appendix 4.B) to each term in the expression for $Y(\omega)$ gives the impulse response

$$y(t) = \frac{1}{3}te^{-t}u(t) - \frac{1}{27}e^{-t}u(t) + \frac{1}{27}e^{-10t}u(t).$$

These are the solutions for versions A and C. For versions B and D everything could be multiplied by a factor of 2.

Problem 4 *(System Composition and Properties)* 22 points

LTI systems $\mathcal{H}_1$ and $\mathcal{H}_2$ with impulse responses $h_1(t)$ and $h_2(t)$ are composed as follows:



(a) (6 Pts) Find $h(t)$, the impulse response of the overall system.

(b) (5 Pts) Suppose that $\mathcal{H}_2$ is not causal, does it imply that $\mathcal{H}$ is not causal? Prove your answer or provide a counter example.

(c) (6 Pts) Compute $h(t)$ when $h_1(t) = \delta(t-6)$ and $h_2(t) = \delta(t+3)$.

Is this system stable? Is it causal? Justify your answer.

(d) (5 Pts) The input to the system in part (c) is given by

$$x(t) = \begin{cases} 1, |t| \le 2 \\ 0, \text{ o.w.} \end{cases}$$

What is the output $y(t)$? Sketch your answer; make sure to clearly label all the axes.

Solution:

(a) As always with these types of problems, it is convenient to define intermediate signals. Let $w(t)$ denote the input to the system $\mathcal{H}_2$. Then

$$w(t) = x(t) + h_1(t) * x(t)$$

and

$$y(t) = w(t) * h_2(t) + x(t).$$

Combining these two equations together we get

$$y(t) = h_1(t) * h_2(t) * x(t) + h_2(t) * x(t) + x(t) = [h_1(t) * h_2(t) + h_2(t) + \delta(t)] * x(t).$$

The overall impulse response is thus

$$h(t) = h_1(t) * h_2(t) + h_2(t) + \delta(t).$$

(b) No. A counter example is $h_1(t) = -\delta(t)$ since then the overall system is $h(t) = \delta(t)$ for any $\mathcal{H}_2$ and this is causal.

(c) For versions A and B:

$$h(t) = h_1(t) * h_2(t) + h_2(t) + \delta(t) = \delta(t-6) * \delta(t+3) + \delta(t+3) + \delta(t) = \delta(t-3) + \delta(t+3) + \delta(t)$$

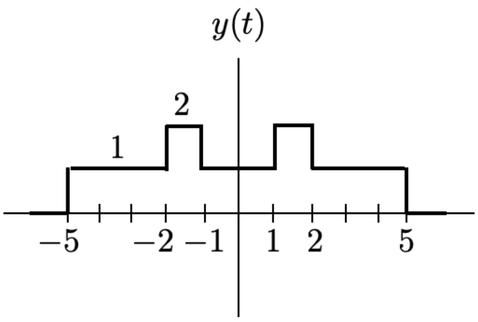
For versions C and D:

$$h(t) = h_1(t) * h_2(t) + h_2(t) + \delta(t) = \delta(t-2) * \delta(t+1) + \delta(t+1) + \delta(t) = \delta(t-1) + \delta(t+1) + \delta(t)$$

In both cases the system is stable but not causal. The system is stable since $x(t) < B$ for all t implies that $y(t) = x(t-3) + x(t+3) + x(t) < 3B$ for t. Thus, bounded input leads to a bounded output. The system is not causal since $y(t) = x(t-3) + x(t+3) + x(t)$ and $y(t_0)$ depends on $x(t_0 - 3)$.

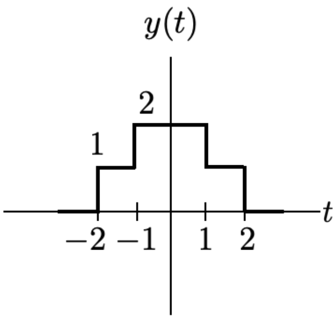
(d) For versions A and B:

$$y(t) = x(t-3) + x(t+3) + x(t)$$



For versions C and D:

$$y(t) = x(t-1) + x(t+1) + x(t)$$



Problem 5 *(Convolution.)* 22 points

During the course we have seen as the impulse response completely characterizes an LTI system. In this problem we define another type of response which we will call here the *'geometric response'* which could also be used to characterize an LTI system. It is defined, in discrete time, as:

$$g[n] = \left(\left(\frac{1}{2}\right)^n u[n]\right) * h[n].$$

That is, $g[n]$ characterizes how an LTI system reacts to the signal $\left(\frac{1}{2}\right)^n u[n]$.

(a) (8 Pts) An LTI system is known to have an impulse response

$$h[n] = \frac{3}{4}u[n-3]$$

find its 'geometric response' $g[n]$.

(b) (8 Pts) An LTI system is known to have the 'geometric response'

$$g[n] = \left(\frac{1}{2}\right)^{n-1} u[n-2].$$

Find the impulse response $h[n]$ of the system.
Hint: there is more than one way to obtain the correct answer. But if you are really stuck, try part (c) first.

Recall that input-output relationship of a stable LTI system could be related by the equation

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

where $Y(e^{j\omega})$ is the DTFT for the output signal $y(t)$, $X(e^{j\omega})$ is the DTFT for the input signal $x(t)$, and $H(e^{j\omega})$ is the DTFT for the impulse response $h(t)$.

(c) (6 Pts) Assume that the DTFT of the 'geometric response' $g[n]$ of an LTI system exists. Find the same input-output relationship in the frequency domain in terms of this DTFT. That is, you need to derive the formula that relates $X(e^{j\omega})$ to $Y(e^{j\omega})$ in terms of $G(e^{j\omega})$.

Solution: (a) We can solve this exercise writing the convolution operation:

$$\begin{aligned} g[n] &= \left(\frac{1}{2^n}u[n]\right) * h[n] \\ &= \frac{3}{4}\sum_{k=-\infty}^{\infty}\frac{1}{2^k}u[k]u[n-3-k] \\ &= \frac{3}{4}\sum_{k=0}^{n-3}\frac{1}{2^k} \\ &= \frac{3}{2}\left(1-\frac{4}{2^n}\right)u[n-3]. \end{aligned}$$

The last equation comes from evaluating the geometric sum for $n \geq 3$ and noticing that the sum is zero for $n < 3$.

This is the solution for versions A and C. For version B and D multiply the result by $\left(\frac{4}{3}\right)^2$.

(b) We can solve this by noticing that:

$$\delta[n] = z[n] - \frac{1}{2}z[n-1]$$

where $z[n] = \left(\frac{1}{2}\right)^n u[n]$.

Then

$$h[n] = \delta[n] * h[n] = z[n] * h[n] - \left(\frac{1}{2}z[n-1]\right) * h[n] = g[n] - \frac{1}{2}g[n-1]$$

and

$$h[n] = \left(\frac{1}{2}\right)^{n-1} u[n-2] - \frac{1}{2}\left(\frac{1}{2}\right)^{n-2} u[n-3] = \left(\frac{1}{2}\right)^{n-1}(u[n-2]-u[n-3]) = \frac{1}{2}\delta[n-2]$$

Another way to solve this problem is to notice that

$$g[n] = \left(\frac{1}{2}\right)^{n-1} u[n-2] = \frac{1}{2}\left(\frac{1}{2}\right)^{n-2} u[n-2] = \frac{1}{2}z[n-2] = z[n] * \left(\frac{1}{2}\delta[n-2]\right)$$

Finally, some of you solved this by writing down the equation

$$\left(\frac{1}{2}\right)^{n-1} u[n-2] = \left(\left(\frac{1}{2}\right)^n u[n]\right) * h[n]$$

and inverting the convolution in the frequency domain.

(c) Given the results of the previous point, we recall:

$$h[n] = g[n] - \frac{g[n-1]}{2}$$

Given the linearity of the Fourier transform and the time-shift property we can write:

$$H(e^{j\omega}) = G(e^{j\omega}) - \frac{e^{-j\omega}}{2}G(e^{j\omega})$$

hence

$$Y(e^{j\omega}) = X(e^{j\omega})G(e^{j\omega})(1 - \frac{e^{-j\omega}}{2})$$